

COXETER ELEMENTS AND ROOT BASES

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ABSTRACT. Let \mathfrak{g} be a Lie algebra of type A, D, E with fixed Cartan subalgebra \mathfrak{h} , root system R and Weyl group W . We show that a choice of Coxeter element $C \in W$ gives a root basis for \mathfrak{g} . Moreover, using the results of [KT] we show that this root basis gives a purely combinatorial construction of \mathfrak{g} , where root vectors correspond to vertices of a certain quiver $\widehat{\Gamma}$, and with respect to this basis the structure constants of the Lie bracket are given by paths in $\widehat{\Gamma}$. This construction is then related to the constructions of Ringel and Peng and Xiao.

1. INTRODUCTION

Let Γ be a Dynkin graph of type A, D, E . Let \mathfrak{g} and $U_q(\mathfrak{g})$ be the corresponding Lie algebra and quantum group respectively. By choosing an orientation Ω of Γ , one obtains a quiver $\vec{\Gamma} = (\Gamma, \Omega)$. Ringel used the category $\text{Rep}(\vec{\Gamma})$ of representations of $\vec{\Gamma}$ to realise \mathfrak{n}_+ and $U_q\mathfrak{n}_+$ (see [R1], [R2]). Peng and Xiao then used a related category, $\mathcal{D}^b \text{Rep}(\vec{\Gamma})/T^2$, to realise the whole Lie algebra \mathfrak{g} . The drawback of these constructions is the necessity of choosing an orientation of the Dynkin diagram.

Motivated by these results and the ideas of Ocneanu [O], the main goal of this paper is to use a Coxeter element, and the results in [KT], to construct a root basis in the Lie algebra \mathfrak{g} and to determine the structure constants of the Lie bracket in purely combinatorial terms.

In [KT] it was shown a choice of Coxeter element gives a bijection between R and a certain quiver $\widehat{\Gamma}$, which identifies roots in R and vertices in $\widehat{\Gamma}$. This bijection then identifies vertices in $\widehat{\Gamma}$ with basis vectors E_α . Using this identification and choice of basis, it is possible to determine the structure constants of the Lie bracket from paths in $\widehat{\Gamma}$. Thus it is possible to realise the Lie algebra \mathfrak{g} completely in terms of the quiver $\widehat{\Gamma}$. This construction is then independent of any choice of orientation of Γ or choice of simple roots.

The case of $U_q(\mathfrak{g})$ for $q \neq 1$, is also of interest. However, a full analysis is the subject of ongoing research.

The main result will now be stated. The proof of this theorem will be left to Section 4 and Section 5. In Section 5 this construction will be related to the constructions of Ringel and Peng-Xiao.

Theorem 1.1. *Let \mathfrak{g} be a Lie algebra of type A, D, E with fixed Cartan subalgebra \mathfrak{h} . This gives a root system R with Weyl group W . Fix a Coxeter element $C \in W$.*

- (1) *The choice of a Coxeter element C gives a root basis $\{E_\alpha\}_{\alpha \in R}$ for \mathfrak{g} .*

(2) Let $\langle \cdot, \cdot \rangle$ be the de-symmetrization of the bilinear form (\cdot, \cdot) given by

$$\langle x, y \rangle = ((1 - C)^{-1}x, y).$$

Then the Lie bracket is given by

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta} & \text{for } \alpha + \beta \in R \\ 0 & \text{for } \alpha + \beta \notin R \text{ and } \alpha \neq -\beta \end{cases}.$$

2. PRELIMINARIES

2.1. Notation. Let \mathfrak{g} be a simple Lie algebra of type A, D, E , and let \mathfrak{h} be a fixed Cartan subalgebra. Denote by R the root system, W the Weyl group, and Γ the Dynkin diagram associated to the pair $(\mathfrak{g}, \mathfrak{h})$. Thus Γ is a Dynkin diagram of type A, D, E .

Let $\Pi = \{\alpha_i\}_{i \in \Gamma}$ denote a set of simple roots.

Since the Weyl group acts simply-transitively on sets of simple roots, there is a unique element which takes Π to $-\Pi$. This element is called the longest element and denoted by w_0 .

For $i \in \Gamma$ define i^\sim by $-\alpha_{i^\sim} = w_0(\alpha_i)$, where $w_0 \in W$ is the longest element.

A set of simple roots Π is compatible with a Coxeter element $C \in W$ if there is a reduced expression $C = s_{i_1} s_{i_2} \cdots s_{i_r}$, where each simple reflection appears exactly once. In other words, Π is compatible with C if $l^\Pi(C) = r$, where l^Π is the length of a reduced expression in terms of the simple reflections s_i .

Let $U_q(\mathfrak{g})$ be the corresponding quantum group. It is generated by elements $E_i, F_i, K_i^{\pm 1}$, where $i \in \Gamma$. In particular, for $q = 1$ this gives the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} .

2.2. The Quiver $\widehat{\Gamma}$. Given the Dynkin diagram Γ of type A, D, E with Coxeter number h , construct a quiver $\widehat{\Gamma} \subset \Gamma \times \mathbb{Z}_{2h}$ as follows:

- (1) Choose a “parity” function $p : \Gamma \rightarrow \mathbb{Z}_2$, so that $p(i) = p(j) + 1$ for i, j connected in Γ .
- (2) Using p , define the vertex set of $\widehat{\Gamma}$ to be $\widehat{\Gamma}_0 = \{(i, n) \mid p(i) + n \equiv 0 \pmod{2}\}$.
- (3) The arrows are given by $(i, n) \rightarrow (j, n+1)$ for i, j connected in Γ .
- (4) Define a “twist” map $\tau : \widehat{\Gamma} \rightarrow \widehat{\Gamma}$ by $\tau(i, n) = (i, n+2)$.

Example 2.1. For the graph $\Gamma = D_5$ the quiver $\widehat{\Gamma}$ is shown in Figure 1. Note that this is the Auslander-Reiten quiver of the category $\mathcal{D}^b(\overrightarrow{\Gamma})/T^2$ for any choice of orientation Ω . Here the \mathbb{Z}_{2h} direction is vertical and the translation acts vertically, while in most of the literature the \mathbb{Z}_{2h} direction is horizontal and the translation acts horizontally to the right.

A function $h : \Gamma \rightarrow \mathbb{Z}_{2h}$ such that $h(i) = h(j) \pm 1$ for i, j connected in Γ will be called a “height function”. Note that such a map defines an orientation on Γ by $i \rightarrow j$ if i, j are connected and $h(j) = h(i) + 1$. This orientation will be denoted by Ω_h . A height function h also gives an embedding of the quiver (Γ, Ω_h) in $\widehat{\Gamma}$, given by $i \mapsto (i, h(i))$. The image of such an embedding is called a “slice” and is denoted by Γ_h .

For a height function h , if $i \in \Gamma$ is a sink or source for Ω_h define a new height function $s_i h$ by

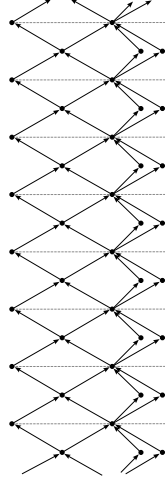


FIGURE 1. The quiver $\widehat{\Gamma}$ for graph $\Gamma = D_5$. Recall that $\widehat{\Gamma}$ is periodic, so the arrows leaving the top level are the same as the incoming arrows at the bottom level.

$$s_i h(j) = \begin{cases} h(i) \pm 2 & \text{if } j = i \text{ where the sign is } + \text{ for } i \text{ a source, } - \text{ for } i \text{ a sink} \\ h(j) & \text{if } j \neq i \end{cases}$$

The orientation determined by $s_i h$ is denoted by $s_i \Omega$ and is obtained by reversing all arrows at i .

Define a function $\langle \cdot, \cdot \rangle_{\widehat{\Gamma}} : \widehat{\Gamma} \times \widehat{\Gamma} \rightarrow \mathbb{Z}$ by setting

$$\begin{aligned} \langle (i, n), (j, n) \rangle_{\widehat{\Gamma}} &= \delta_{ij}, \\ \langle (i, n), (j, n+1) \rangle_{\widehat{\Gamma}} &= \text{the number of paths } (i, n) \rightarrow \cdots \rightarrow (j, n+1) \\ &= \text{the number of edges between } i, j \text{ in } \Gamma. \end{aligned}$$

Then for any $q = (k, m) \in \widehat{\Gamma}$ use the relation

$$\langle q, (i, n) \rangle_{\widehat{\Gamma}} - \sum_{j \sim i} \langle q, (j, n+1) \rangle_{\widehat{\Gamma}} + \langle q, (i, n+2) \rangle_{\widehat{\Gamma}} = 0$$

for i, j connected in Γ , to extend the definition.

It was shown in [KT] (Proposition 7.4) that this function is well-defined.

Given a Coxeter element $C \in W$, it was shown in [KT] that there is a bijection $R \rightarrow \widehat{\Gamma}$ with the following properties:

- (1) It identifies the Coxeter element C with the “twist” $\tau : \widehat{\Gamma} \rightarrow \widehat{\Gamma}$.
- (2) It gives a bijection between simple systems Π , compatible with C , and height functions $h : \Gamma \rightarrow \widehat{\Gamma}$.
- (3) For each height function h one obtains an explicit description of the corresponding positive roots and negative roots as disjoint connected subquivers of $\widehat{\Gamma}$, as well as a

reduced expression for the longest element w_0 in the Weyl group. The reduced expression for w_0 is given as a sequence of source to sink reflections taking the slice $\Gamma_{h\Pi}$ to the slice $\Gamma_{h-\Pi}$.

- (4) There is a de-symmetrization of the inner product on R , denoted by $\langle \cdot, \cdot \rangle$ which is analogous to the Euler form in quiver theory. Moreover, under the bijection Φ , this form is identified with $\langle \cdot, \cdot \rangle_{\hat{\Gamma}}$ in $\hat{\Gamma}$.

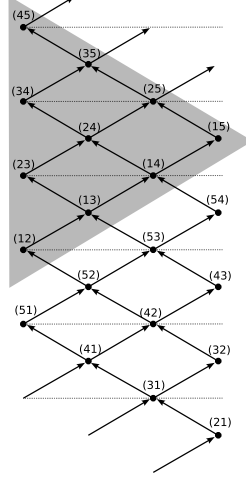


FIGURE 2. The bijection $\Phi : R \rightarrow \hat{\Gamma}$ for $\Gamma = A_4$. For each vertex in $\hat{\Gamma}$ the corresponding root $\alpha \in R$ is shown. The notation (ij) stands for $e_i - e_j$. The set of positive roots corresponding to Π is shaded. Recall that $\hat{\Gamma}$ is periodic, so that arrows leaving the top level are identified with the incoming arrows on the bottom level.

Example 2.2. For $\Gamma = A_4$ with $\Pi = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5\}$ and $C = s_1 s_2 s_3 s_4$ the bijection $R \rightarrow \hat{\Gamma}$ is given in Figure 2.

3. BRAID GROUP ACTION

In this section the definition and relevant results of the braid group operators as defined in [J] are reviewed. For more details see [J], or [L].

Fix a system of simple roots Π . Let $E_i, F_i, K_i^{\pm 1}$ denote the corresponding generators of $U_q(\mathfrak{g})$. For simple roots α_i define operators T_i, T'_i on any finite dimensional module V by setting for $v \in V_\lambda$:

$$T_i(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{b-ac} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

$$T'_i(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{ac-b} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

with $m = (\lambda, \alpha_i^\vee)$.

Then there are unique automorphisms of $U_q(\mathfrak{g})$, also denoted by T_i, T'_i so that for any $u \in U_q(\mathfrak{g})$ and $v \in V$ we have $T_i(uv) = T_i(u)T_i(v)$. The operator T_i acts on weights by the reflection s_i .

The automorphisms T_i satisfy the braid relations:

$$T_i T_j = T_j T_i \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_i T_j T_i = T_j T_i T_j \quad \text{for } (\alpha_i, \alpha_j) = -1$$

For the automorphism T_i there are the following formulae:

$$T_i E_i = -F_i K_i$$

$$T_i F_i = -K_i^{-1} E_i$$

$$T_i E_j = E_j \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_i E_j = E_i E_j - q^{-1} E_j E_i \quad \text{for } (\alpha_i, \alpha_j) = -1$$

$$T_i F_j = F_j \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_i F_j = F_i F_j - q^{-1} F_j F_i \quad \text{for } (\alpha_i, \alpha_j) = -1$$

In fact, there are automorphisms T_α for any root α . As above, define T_α on a module V by setting for $v \in V_\lambda$:

$$T_i(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{b-ac} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

$$T'_i(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{ac-b} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

where $E_\alpha, F_\alpha \in U_q(\mathfrak{g})$ satisfy the $U_q(\mathfrak{sl}_2)$ relations and $m = (\lambda, \alpha^\vee)$.

Lemma 3.1. *Let Φ be an automorphism of $U_q(\mathfrak{g})$ such that $E_\alpha = \Phi(E_i)$ and $F_\alpha = \Phi(F_i)$. Then $T_\alpha = \Phi T_i \Phi^{-1}$.*

The automorphisms T_α satisfy relations similar to those of the T_i :

$$T_\alpha E_\alpha = -F_\alpha K_\alpha$$

$$T_\alpha F_\alpha = -K_\alpha^{-1} E_\alpha$$

$$T_\alpha E_\beta = E_\beta \quad \text{for } (\alpha, \beta) = 0$$

$$T_\alpha E_\beta = E_\alpha E_\beta - q^{-1} E_\beta E_\alpha \quad \text{for } (\alpha, \beta) = -1$$

$$T_\alpha F_\beta = F_\beta \quad \text{for } (\alpha, \beta) = 0$$

$$T_\alpha F_\beta = F_\alpha F_\beta - q^{-1} F_\beta F_\alpha \quad \text{for } (\alpha, \beta) = -1$$

Since the operators T_i satisfy the braid relations it is possible to define an operator T_w for any $w \in W$. For any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for $w \in W$ define $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$.

The following Lemma will be useful. It can be found in [J], Proposition 8.20.

Lemma 3.2. *If $w \in W$ is such that $w(\alpha_i) \in R_+$, then $T_w(E_i) \in U_q^+$. If, in addition, $w(\alpha_i) = \alpha_j$, then $T_w(E_i) = E_j$.*

For the case to be considered in the following sections, this result gives the following important Corollary.

Corollary 3.3. *Let $w_0 \in W$ be the longest element. Then $T_{w_0}(E_i \cdot) = T_i E_i = -F_i K_i$.*

Proof. Let $w_0 = s_i w$ be a reduced expression for w_0 , so that $T_{w_0} = T_i T_w$. Then since

$$w(\alpha_i \cdot) = s_i w_0(\alpha_i \cdot) = s_i(-\alpha_i) = \alpha_i$$

the Lemma gives $T_w(E_i \cdot) = E_i$, and the result follows by applying T_i . \square

4. LONGEST ELEMENT AND CONSTRUCTION OF ROOT VECTORS

Let Π be a simple system, and let $R = R_+ \cup R_-$ be the corresponding polarization. Let w_0 be the longest element. A reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$ is said to be *adapted* to an orientation Ω of Γ if i_k is a source for $s_{i_1} \cdots s_{i_{k-1}} \Omega$. In particular i_1 is a source of Ω .

Lemma 4.1. *Given any orientation Ω , there is a reduced expression adapted to Ω , and moreover, any two expressions adapted to Ω are related by $s_i s_j = s_j s_i$ with $n_{ij} = 0$.*

Proof. Recall that any height function h determines an orientation Ω_h and that for any orientation Ω there is a choice of h so that $\Omega = \Omega_h$. So take some h corresponding to Ω . Note that any reduced expression adapted to Ω gives a sequence of source to sink moves taking the slice Γ_h to the slice Γ_{-h} where Γ_{-h} is the slice corresponding to the simple roots $-\Pi$.

Let $w_0 = s_{i_1} \cdots s_{i_l}$ and $w_0 = s_{i'_1} \cdots s_{i'_l}$ be two different reduced expressions adapted to Ω . Let k be the first index where they differ. Write $i_k = i$ and $i'_k = j$ to simplify notation. Then there are reduced expressions

$$w_0 = w s_i w_1 s_j w_2$$

$$w_0 = w s_j w'_1 s_i w'_2$$

where s_j does not appear in w_1 and s_i does not appear in w'_1 . Thus i, j are both sources for $w\Omega$ and hence $n_{ij} = 0$. Note that since w_1 is obtained as a sequence of source to sink reflections, and since s_j does not appear in w_1 , j remains a source during this process. Hence if s_k appears in w_1 then k is not adjacent to j , so that $n_{jk} = 0$. Thus $w_1 s_j = s_j w_1$, which gives:

$$w_0 = w s_i w_1 s_j w_2$$

$$= w s_i s_j w_1 w_2$$

$$= w s_j s_i w_1 w_2$$

So it is possible to make the two reduced expressions agree at the index k using only the relation $s_i s_k = s_k s_i$ for $n_{ik} = 0$. Continuing in this fashion it is possible to make the expressions agree at every index using only this relation. \square

It is well known that a reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$, adapted to Ω , gives an ordering of the positive roots $R = \{\gamma_1, \dots, \gamma_l\}$ by setting $\gamma_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)$. Such an expression also gives root vectors E_α, F_α for $\alpha \in R_+$ as follows:

$$(4.1) \quad E_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k})$$

$$(4.2) \quad F_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$$

Note that since the T_i satisfy the braid relation, Lemma 4.1 implies that the root vectors defined this way do not depend on the choice of reduced expression adapted to Ω .

Note that if $\gamma_k = s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}$ then $\gamma_k = s_{\gamma_{k-1}} \cdots s_{\gamma_1}\alpha_{i_k}$, and the longest element can be expressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_1}$.

Since $T_{\gamma_k} = (T_{i_1} \cdots T_{i_{k-1}})T_{i_k}(T_{i_1} \cdots T_{i_{k-1}})^{-1}$, then as for reflections,

$$T_{i_1} \cdots T_{i_k} = T_{\gamma_k} \cdots T_{\gamma_1}$$

so the root vectors E_{γ_k} given in Equation 4.1 can be expressed as

$$(4.3) \quad E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_1} E_{i_k}$$

Definition 4.2. Let Π be a set of simple roots, and Ω be an orientation of Γ and let $w_0 = s_{i_1} \cdots s_{i_l}$ be a reduced expression adapted to Ω . A root basis $\{E_\alpha\}_{\alpha \in R}$ is said to be adapted to the pair (Π, Ω) if for $\alpha \in R_+$ the vector E_α is given by Equation 4.1, or equivalently by Equation 4.3.

4.1. Change of Orientation. For a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$, adapted to Ω , define a new reduced expression $w_0 = s_{i_2} \cdots s_{i_l} s_{i_1}$ which is adapted to $s_1\Omega$. Then this gives a new enumeration of positive roots $\{\gamma'_1, \dots, \gamma'_l\}$, and a new collection of root vectors:

$$(4.4) \quad \begin{aligned} \gamma'_1 &= s_{i_1}(\gamma_{i_2}), \gamma'_2 = s_{i_1}(\gamma_{i_3}), \dots, \gamma'_l = \alpha_{i_1} \\ E_{\gamma'} &= T_i^{-1}(E_{s_i \gamma}) \quad \text{for } \gamma \neq \alpha_{i_1} \end{aligned}$$

4.2. Coxeter Element. Now consider the case where there is a fixed Coxeter element $C \in W$ and hence an identification $R \rightarrow \hat{\Gamma}$ as in [KT]. In this case a choice of height function h is identified with a set of simple roots Π compatible with C , and hence determines a polarization $R = R_+^h \cup R_-^h$. A height function also determines a reduced expression for w_0 adapted to the orientation Ω_h . This expression is obtained from $\hat{\Gamma}$ as a sequence of source to sink reflections which take the slice $\Gamma_{h\Pi}$ to the slice $\Gamma_{h-\Pi}$.

Using this reduced expression, there is an associated ordering of the positive roots which gives a completion of the partial order given by paths in $\hat{\Gamma}$. Note that the completion depends on the reduced expression.

Now choose a height function h . Then using the reduced expression for w_0 obtained above, it is possible to define a collection of root vectors E_α for $\alpha \in R_+^h$ using Equation 4.1.

Under the identification $R \rightarrow \hat{\Gamma}$ suppose that $\alpha = (i, n)$, then $C\alpha = (i, n+2)$. For j connected to i , denote by γ_j the root corresponding to vertex $(j, n+1)$. The collection of roots $\{\alpha, \gamma_j, C\alpha\}$ is said to satisfy the fundamental relation in $\hat{\Gamma}$. Such a collection is depicted in Figure 3.

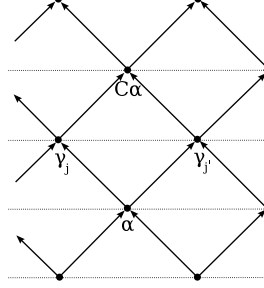


FIGURE 3. A collection of roots $\alpha, \gamma_j, C\alpha \in \widehat{\Gamma}$ satisfying the fundamental relation.

Lemma 4.3. *Let $\alpha, \gamma_j, C(\alpha) \in R_+^h$ satisfy the fundamental relation in $\widehat{\Gamma}$. Then the corresponding root vectors satisfy:*

$$(4.5) \quad E_{C(\alpha)} = \left(\prod_{j-i} T_{\gamma_j} \right) T_{\alpha}(E_{\alpha})$$

Proof. Let h be a fixed height function and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ denote the corresponding set of simple roots and s_i the corresponding simple reflections.

Let $\alpha = (i, n), \gamma_j = (j, n+1), C\alpha = (i, n+2)$ and

$$w_0 = w s_i \left(\prod_{j-i} s_j \right) s_i w'$$

a reduced expression adapted to Ω_h .

Then

$$\begin{aligned} E_{\alpha} &= T_w E_{\alpha_i} \text{ and } T_{\alpha} = T_w T_{\alpha_i} T_w^{-1} \\ E_{\gamma_j} &= T_w T_{\alpha_j} E_{\alpha_j} \text{ and } T_{\gamma_j} = T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1} \\ E_{C\alpha} &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j} \right) E_{\alpha_i} \end{aligned}$$

where the product \prod_{j-i} is taken over all j connected to i in Γ .

On the other hand, using the first two formulae, and comparing with the third one obtains:

$$\begin{aligned} \left(\prod_{j-i} T_{\gamma_j} \right) T_{\alpha} E_{\alpha} &= \prod_{j-i} (T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1}) T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\ &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j} \right) T_{\alpha_i}^{-1} T_w^{-1} T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\ &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j} \right) E_{\alpha_i} \\ &= E_{C\alpha} \end{aligned}$$

□

Now fix a height function h , and hence a choice of compatible simple roots Π_h , an orientation Ω_h , a reduced expression w_0 and a slice $\Gamma_h \subset \widehat{\Gamma}$. Define a root basis as follows:

- (1) For $\beta \in \Gamma_h$ choose $E_\beta \in \mathfrak{g}_\beta$.
- (2) Since any root is of the form $\alpha = C^k \beta$ for some k and some β , define E_α inductively using Equation 4.5, beginning with $E_{C\beta}$ for β a source in Γ_h .

Note that for $C^h \beta = \beta$ this procedure produces another root vector $E'_\beta \in \mathfrak{g}_\beta$.

Proposition 4.4. *Let E_α, E'_α be the root vectors defined above.*

- (1) For $q = 1$ $E'_\alpha = E_\alpha$, so this procedure produces a consistent root basis in \mathfrak{g} .
- (2) For $q \neq 1$ $E'_\alpha = K_\alpha^{-1} E_\alpha K_\alpha$.

Corollary 4.5. *Let $U_q \mathfrak{g}$ denote the corresponding quantum group. For $q \neq 1$ there is a \mathbb{Z} -torsor of vectors $\{E_\alpha^k\}$ for each root α that are related by $E_\alpha^{k+n} = K_\alpha^n E_\alpha^k K_\alpha^{-n}$.*

Proof. To simplify notation, set $E_{\alpha_i} = E_i, F_{\alpha_i} = F_i, K_{\alpha_i} = K_i$. Then using Corollary 3.3 one obtains:

$$\begin{aligned} E'_i &= T_{w_0}(T_{w_0} E_i) \\ &= T_{w_0}(-F_i \cdot K_i \cdot) \\ &= -(T_{w_0} F_i \cdot)(T_{w_0} K_i \cdot) \\ &= -(-K_i^{-1} E_i)(K_i) \\ &= K_i^{-1} E_i K_i \end{aligned}$$

This proves the second part, and to get the first part set $q = 1$ so that $K_i = 1$. \square

Theorem 4.6. *Let h be any height function and denote the associated simple roots and orientation by Π_h and Ω_h respectively.*

- (1) *The root basis defined above is adapted to the pair (Π_h, Ω_h) .*
- (2) *For this choice of root basis the Lie bracket is given by:*

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta} & \text{for } \alpha + \beta \in R \\ 0 & \text{for } \alpha + \beta \notin R \text{ and } \alpha \neq -\beta \end{cases}$$

Proof. Let h be the height function used to construct the root basis $\{E_\alpha\}$. By construction this basis is adapted to the pair (Π_h, Ω_h) . So it is enough to check that if $\{E_\alpha\}$ is adapted to (Π, Ω) , and i is a source for Ω , then $\{E_\alpha\}$ is also adapted to $(s_i \Pi, s_i \Omega)$.

Suppose that $\{E_\alpha\}$ is adapted to (Π, Ω) and that i is a source. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Then since i is a source and w_0 is adapted to Ω , the corresponding reduced expression for the longest element has the form $w_0 = s_i s_{i_2} \cdots s_{i_l}$. By writing $\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$, the longest element can be reexpressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_2} s_{\alpha_i}$. (Note that since i is a source, $\gamma_1 = \alpha_i$.)

Then since $\{E_\alpha\}$ is adapted it is possible to write

$$E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_k}.$$

Now consider the pair $(s_i\Pi, s_i\Omega)$. Denote the simple roots by $\alpha'_j = s_i\alpha_j$ and the corresponding simple reflections $s'_j = s_i s_j s_i$. Then the corresponding reduced expression for the longest element is $w_0 = s'_{i_2} \cdots s'_{i_l} s_i$ and as before if $\gamma'_k = s'_{i_2} \cdots s'_{i_{k-1}}(\alpha'_{i_k})$, then $\gamma'_k = \gamma_{k+1}$ for $k+1 \neq l$ and $\gamma'_l = -\alpha_i$.

Now, if $k+1 \neq l$ then

$$\begin{aligned} E'_{\gamma'_k} &= E_{\gamma_{k+1}} \\ &= T_{\gamma_k} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_{i_{k+1}}} \\ &= T_{\gamma_k} \cdots T_{\gamma_2} E_{s_i \alpha_{i_{k+1}}} \text{ by Equation 4.4} \\ &= T_{\gamma'_{k-1}} \cdots T_{\gamma'_1} E_{\alpha'_{i_k}} \end{aligned}$$

so $E'_{\gamma'_k}$ is given by Equation 4.3.

If $k+1 = l$ then,

$$\begin{aligned} E'_{\gamma'_l} &= E_{-\alpha_i} \\ &= T_{w_0}(E_{\alpha_i}) \\ &= T_{\gamma_l} \cdots T_{\gamma_2} T_{\alpha_i}(E_{\alpha_i}) \\ &= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{s_i \alpha_i} \\ &= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{\alpha'_i} \end{aligned}$$

so again $E'_{\gamma'_k}$ is given by Equation 4.3. Hence $\{E_\alpha\}$ is adapted to the pair $(s_i\Pi, s_i\Omega)$.

The proof of the second part will follow from Corollary 5.4. \square

Note that Proposition 4.4 and Proposition 4.6 prove the main result, Theorem 1.1.

Define $T_C = T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_r}}$, for some choice of compatible simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$, with $C = s_{i_1} s_{i_2} \cdots s_{i_r}$. Since the T_α satisfy the braid relations, the operator does not depend on the choice of compatible simple roots Π .

Proposition 4.7. *The root vectors E_α satisfy $T_C E_\alpha = E_{C\alpha}$.*

5. RINGEL-HALL ALGEBRAS

In this section Ringel and Peng and Xiao's approaches to constructing the Lie algebra \mathfrak{g} from quiver theory is reviewed. This is then related to the construction given in the previous section. For more details on Ringel's construction see [R1], [R2], [DX]. For more details on Peng and Xiao's construction see [PX1] and [PX2].

Let Ω be a fixed orientation of Γ and denote by $\vec{\Gamma} = (\Gamma, \Omega)$ the corresponding quiver. Fix \mathbb{K} , a finite field of order p . Let $\text{Rep}(\vec{\Gamma})$ be the category of representations of this quiver over the field \mathbb{K} , and denote by \mathcal{K} its Grothendieck group. Denote by $\text{Ind} \subset \mathcal{K}$ the set of classes of indecomposable objects. Then Gabriel's Theorem gives an identification $\text{Ind} \rightarrow R_+$ between indecomposable objects and positive roots of the corresponding root system. Moreover, if $\langle \cdot, \cdot \rangle$ is defined on \mathcal{K} by $\langle X, Y \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y)$, then the form given by $\langle X, Y \rangle = \langle X, Y \rangle + \langle Y, X \rangle$ is identified with the bilinear form on the root lattice. The form $\langle \cdot, \cdot \rangle$ is called the Euler form.

Ringel then constructed an associative algebra $(\mathcal{H}_p, *)$ as follows:

- (1) As a vector space \mathcal{H}_p is spanned by $[M] \in \mathcal{K}$.
- (2) For objects M, N, L define $F_{M,N}^L = |\{X \subset L \mid X \simeq M \text{ and } L/X \simeq N\}|$. (Since \mathbb{K} is finite, this number is well-defined.)
- (3) Define an operation $*$ on \mathcal{H}_p by the formula $[M] * [N] = \sum_{[L]} F_{M,N}^L [L]$.

The following Theorem summarizes the main results of Ringel.

Theorem 5.1. *Let $(\mathcal{H}_p, *)$ be the algebra defined above.*

- (1) For $n = (n_\alpha) \in (\mathbb{Z}_+)^{R_+}$ set $M_n = \bigoplus n_\alpha M_\alpha$ where M_α is the indecomposable corresponding to root α . Then $\{[M_n]\}$ is a PBW-type basis of the algebra \mathcal{H}_p , so that all structure constants $F_{[M],[N]}$ are in $\mathbb{Z}[p]$. Hence the Hall algebra can be considered with p as a formal parameter. After making the substitution $q = p^{1/2}$, \mathcal{H}_q can be identified with $U_q \mathfrak{n}_+$.
- (2) For $q = 1$, this gives an isomorphism $\Psi : U \mathfrak{n}_+ \rightarrow \mathcal{H}_1$ which is given by $E_\alpha \mapsto [M_\alpha]$, where M_α denotes the indecomposable representation of $\vec{\Gamma}$ corresponding to root α . In particular the set $\{[M_\alpha]\}$ is a root basis for \mathfrak{n}_+ .
- (3) In the case $q = 1$, the Lie bracket $[\cdot, \cdot]$ is given by $[M_\alpha, M_\beta] = (-1)^{\langle M_\alpha, M_\beta \rangle} M_{\alpha+\beta}$ for $\alpha + \beta \in R$. Here $\langle \cdot, \cdot \rangle$ is the Euler form.

The polynomial $F_{M,N}^L(p)$ appearing in Part 1 of the Theorem is called the ‘‘Hall polynomial’’.

As mentioned before, Peng and Xiao extended the results of Ringel to obtain a description for all of \mathfrak{g} . This construction is briefly recalled here. For a more details see [PX1], [PX2].

Peng and Xiao considered the ‘‘root category’’, $\mathcal{D} = \mathcal{D}^b(\vec{\Gamma})/T^2$, so that indecomposable objects are in bijection with all roots. If $M \in \text{Rep}(\vec{\Gamma})$ is indecomposable, then considering this as a complex concentrated in degree 0, M is also indecomposable in \mathcal{D} . These objects correspond to positive roots, while their translates, TM , correspond to negative roots. (Up to isomorphism, this is a full description of indecomposable objects in \mathcal{D} .) Denote by M_α the class of indecomposable corresponding to root $\alpha \in R_+$. Peng and Xiao then constructed a Lie algebra $\mathcal{H}_\mathcal{D}$ as follows:

- (1) Set $\mathcal{H}_\mathcal{D} = \mathfrak{N} \oplus \mathfrak{H}$ where $\mathfrak{H} = \mathcal{K}(\mathcal{D})$ and \mathfrak{N} is the free abelian group with basis $\{u_{[M]}\}$ indexed by isomorphism classes of objects $[M]$.
- (2) Let $h_M = [M] \in \mathcal{K}(\mathcal{D})$.
- (3) Define a bilinear operation $[\cdot, \cdot]$ on $\mathcal{H}_\mathcal{D}$ by:
 - (a) $[\mathfrak{H}, \mathfrak{H}] = 0$
 - (b) $[u_M, u_N] = \sum_{[L]} (F_{M,N}^L(1) - F_{N,M}^L(1)) u_L$, for $N \neq TM$, where $F_{M,N}^L$ is the Hall polynomial.
 - (c) $[u_M, u_{TM}] = \frac{h_M}{d(M)}$ where $d(M) = \dim \text{End}(M)$
 - (d) $[h_M, u_N] = -(M, N)_\mathcal{D} u_N = -[u_N, h_M]$ where $(\cdot, \cdot)_\mathcal{D}$ is the symmetrized Euler form on $\mathcal{K}(\mathcal{D})$.
- (4) For $\alpha \in R_+$, let $h_\alpha = h_M$ where $\dim M_\alpha = \alpha$.
- (5) For $\alpha \in R_+$, let $M_\alpha = [M_\alpha]$ where $\dim M_\alpha = \alpha$.
- (6) For $\alpha \in R_+$ let $M_{-\alpha} = -[TM_\alpha]$ where $\dim M_\alpha = \alpha$.

Theorem 5.2. *Let $(\mathcal{H}_\mathcal{D}, [\cdot, \cdot])$ be defined as above.*

- (1) $(\mathcal{H}_{\mathcal{D}}, [\cdot, \cdot])$ is a Lie algebra.
- (2) The collection $\{M_{\alpha}, M_{-\alpha}\}_{\alpha \in R_+}$ defined above is a root basis for $\mathcal{H}_{\mathcal{D}}$.
- (3) The map given by $E_{\alpha} \mapsto M_{\alpha}$, $F_{\alpha} \mapsto M_{-\alpha}$ and $H_{\alpha} \mapsto h_{\alpha}$ for $\alpha \in R_+$ induces an isomorphism of Lie algebras $\mathfrak{g} \rightarrow \mathcal{H}_{\mathcal{D}}$. Hence $\mathcal{H}_{\mathcal{D}}$ can be identified with the \mathbb{Z} -form of \mathfrak{g} .

For details see [PX1] Section 4.

Recall that given a height function h , there is a corresponding set of simple roots Π_h and a polarization $R = R_+^h \cup R_-^h$. Let E_{α} be the root vectors defined in Section 4. Define a triangular decomposition $\mathfrak{g} = \mathfrak{n}_-^h \oplus \mathfrak{h} \oplus \mathfrak{n}_+^h$ by setting $\mathfrak{n}_{\pm}^h = \langle E_{\alpha} \rangle_{\alpha \in R_{\pm}^h}$.

A height function h also gives an orientation Ω_h of Γ and hence a quiver $\vec{\Gamma} = (\Gamma, \Omega_h)$. As above, denote by \mathcal{K} the corresponding Grothendieck group, and by $\text{Ind} \subset \mathcal{K}$ the set of indecomposable classes in \mathcal{K} . Then there is a bijection $R_+^h \rightarrow \text{Ind}$, given by $\alpha \mapsto [M_{\alpha}]$.

Proposition 5.3. *Let h be a height function. Then the identification $R_+^h \rightarrow \text{Ind}$ in $\text{Rep}(\vec{\Gamma})$ induces an isomorphism $U\mathfrak{n}_+^h \rightarrow \mathcal{H}_1$ given by $E_{\alpha} \mapsto [M_{\alpha}]$. Moreover, the identification $R \rightarrow \text{Ind}(\mathcal{D})$ in the root category \mathcal{D} gives an isomorphism $\mathfrak{g} - \mathfrak{h} \rightarrow \mathcal{H}_{\mathcal{D}} - \mathfrak{H}$, given by $E_{\alpha} \mapsto [M_{\alpha}]$, $E_{-\alpha} \mapsto -[TM_{\alpha}]$ for $\alpha \in R_+^h$.*

Corollary 5.4. *The Lie algebra \mathfrak{g} can be realised combinatorially in terms of $\hat{\Gamma}$: It has root basis E_{α} for $\alpha \in \hat{\Gamma}$ and Lie bracket given by*

$$(5.1) \quad [E_{\alpha}, E_{\beta}] = \begin{cases} (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta} & \text{for } \alpha + \beta \in R \\ 0 & \text{for } \alpha + \beta \notin R \text{ and } \alpha \neq -\beta \end{cases}$$

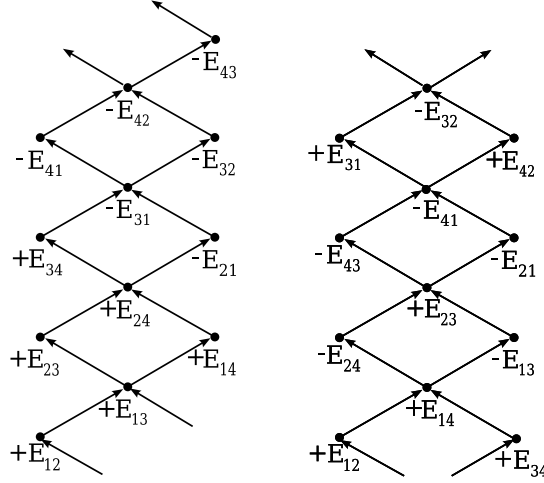


FIGURE 4. Two different root bases for \mathfrak{sl}_4 coming from different choices of C . The case $C = (1234)$ is shown in the figure to the left. The case $C = (1243)$ is shown in the figure to the right. For each vertex in $\hat{\Gamma}$ the corresponding root vector E_{α} is shown in terms of the matrix units E_{ij} . Recall that $\vec{\Gamma}$ is periodic, so that arrows leaving the top level are identified with the incoming arrows on the bottom level.

Proof. The only thing to be checked is that in terms of the E_α constructed in Section 4, the structure constants of the Lie bracket are given by Equation 5.1. For $\alpha, \beta \in R$ with $\alpha \neq -\beta$ there is a choice of compatible simple roots Π so that $\alpha, \beta \in R_+^\Pi$. Let h be the corresponding height function. Then by Proposition 5.3 the identification $Un_+^h \simeq \mathcal{H}_1$ gives that

$$[E_\alpha, E_\beta] = [M_\alpha, M_\beta] = (-1)^{\langle \alpha, \beta \rangle} M_{\alpha+\beta} = (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta}.$$

□

Remark 5.5. Note that the “Euler cocycle” $(-1)^{\langle \cdot, \cdot \rangle}$, defines a cohomologous cocycle, and hence the same extension, as in the construction of \mathfrak{g} given in [FLM].

Example 5.6. Consider the case $\Gamma = A_3$, so that $\mathfrak{g} = \mathfrak{sl}_4$. Let \mathfrak{h} be the diagonal matrices. Then the roots are $\alpha = e_i - e_j$ for $i \neq j$, where $e_k(h) = h_{kk}$ for $h \in \mathfrak{h}$. The root space corresponding to root $e_i - e_j$ is $\mathbb{C}E_{ij}$, where E_{ij} is the corresponding matrix unit. For two different choices of Coxeter element C , two different root bases are shown in Figure 4. In each case the Lie bracket is then given by the Equation 5.1 and the form $\langle \cdot, \cdot \rangle$ can be computed explicitly in terms of $\hat{\Gamma}$.

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